Riemann Integral

$B[a,b]$ consists of all bounded real-valued functions on $[a,b]$

$\mathcal{P}_a[a,b]$ " " " partitions of $[a,b]$. Let $f \in B[a,b]$ & $P \in \mathcal{P}_a[a,b]$:

$$P := \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\} \text{ with some } n \in \mathbb{N},$$

$$I_i := [x_{i-1}, x_i] \quad (i = 1, \cdots, n) \quad \ell(I_i) := x_i - x_{i-1}$$

$$M_i := \sup\{f(x) : x \in I_i\}, \quad m_i := \inf\{f(x) : x \in I_i\}$$

$$u(f;P) := \sum_{i=1}^{n} m_i \cdot \ell(I_i), \quad U(f;P) := \sum_{i=1}^{n} M_i \cdot \ell(I_i)$$

$$\underline{\int} f := \sup_{P \in \mathcal{P}_a[a,b]} u(f;P), \quad \overline{\int} f := \inf_{P \in \mathcal{P}_a[a,b]} U(f;P)$$

Show that

1. $u(P) \left(:= u(f;P) \right)$ is $\uparrow_P$: partitions $P \subseteq P' \Rightarrow u(P) \leq u(P')$

   $U(P) \downarrow_P$

2. $f \mapsto \overline{\int} f$ is sublinear on $B[a,b]$:

$$\overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g \quad \forall f, g \in B[a,b]$$

$$\overline{\int}(\alpha f) = \alpha \overline{\int} f \quad \forall f \in B[a,b] \;\&\; \forall\, 0 \leq \alpha \in \mathbb{R}$$

3. $f \mapsto \underline{\int} f$ is "superlinear" ("$\leq$" to be replaced by "$\geq$" in Q2)

   & $\underline{\int}(-f) = -\overline{\int} f \quad \forall f \in B[a,b]$

4. Let $\mathcal{R}[a,b] := \{f \in B[a,b] : \underline{\int} f = \overline{\int} f\}$ and define

$$\int_a^b f = \underline{\int} f = \overline{\int} f \quad \forall f \in \mathcal{R}[a,b].$$

   Then $f \mapsto \int_a^b f$ is linear

5. Let $\mathcal{P}a[a,b] \ni P \subseteq P'$ & $\#(P' \setminus P) = N \in \mathbb{N}$ (i.e. P' is obtained from P by adding N many partition points). Then $0 \le U(P) - U(P') \le N(M-m)\|P\|$ where $\|P\| := $ largest subinterval-length of P and $f : [a,b] \to [m,M]$. (Hint: Suff(??) to consider $N=1$).

6. Show that

$$\bar{\int} f = \lim_{P} U(f;P) = \lim_{\|P\| \to 0} U(f;P)$$

where the two limits are respectively in the following sense:

1) $\forall \varepsilon > 0 \; \exists P_\varepsilon \in \mathcal{P}a[a,b]$ s.t.

$$\left| U(f;P) - \bar{\int} f \right| < \varepsilon \text{ whenever } P_\varepsilon \subseteq P \in \mathcal{P}a[a,b]$$

2) $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$\left| U(f;P) - \bar{\int} f \right| < \varepsilon \text{ whenever } \|P\| < \delta.$$

7. Do similarly for $\underline{\int} f$ and $u(f;P)$.

8. Let $f \in B[a,b]$, $\alpha \in \mathbb{R}$. Show that $f \in \mathcal{R}[a,b]$ with $\int_a^b f = \alpha$ if and only if $\forall \varepsilon > 0 \; \exists P_\varepsilon \in \mathcal{P}a[a,b]$ s.t.

$$\left| \alpha - \sum_{i=1}^{n} f(\xi_i)\, \ell(I_i) \right| < \varepsilon \quad \forall \xi_i \in I_i \; (i = 1, 2 \ldots n)$$

whenever $P_\varepsilon \subseteq P \in \mathcal{P}a[a,b]$ ($I_1, I_2, \ldots I_n$ are the subintervals of P).

9. Using the other limit formulation (in terms of $\|P\| \to 0$) in Q6, do a corresponding one similar to Q8.